his illustrious brother and survived him by nine years. He was well and favorably known both as a mathematician and as a hydrographer. He was attached to the ill-fated voyage of discovery led by La Pérouse (La Peyrouse), but fortunately was compelled to abandon the trip owing to illness which developed at Teneriffe. He became professor of mathematics in the Ecole militaire, and later became professor and examiner in hydrography in the navy. Thanks to his brother's influence, the latter having a position as minister in the revolutionary period, he passed safely through the Days of Terror, and in 1797, when the worst was over, he occupied the position which the letter indicates.

QUESTIONS AND DISCUSSIONS.

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DISCUSSIONS.

The discussion by Professor Moritz on geometric illustrations of indeterminate forms should prove interesting and valuable. It is not superfluous to take this occasion to point out the extraordinary misunderstanding and logical inaccuracy often associated with the subject of indeterminate forms. Errors are common in our textbooks, and some teachers have never become acquainted with the correct point of view. It will be necessary first to touch upon the ideas of zero and infinity, which are scarcely less hazy.

In the first place, it should be clearly understood that the number system which is agreed upon by mathematicians as best representing the practical needs of analysis—irrespective of the obvious possibility of other self-consistent number systems, whose properties may agree to a greater or less extent with those of the usual system—does contain a number *zero*, as real, with as definite a meaning as any other number, and having, with one exception, properties like those of any other number. This ordinary number system, on the other hand, does *not* contain a number *infinity*. If any one prefers a system which shall not contain zero or which shall contain infinity, then, so long as he formulates self-consistent laws of operation, he is entirely free to use his system; but he cannot expect mathematicians generally to follow his lead unless he can show that his system involves fewer exceptions or inconveniences than that to which they are accustomed.

If we are to use the ordinary system, we shall have no difficulty provided we take account of the fact that its laws are uniform and general, except in one single particular: the number zero cannot be used as a divisor. The expression 2/0 does not represent some mysterious unattainable large number; it represents no number whatever. The expression 0/0 does not represent several, many, or all numbers[1]; it represents no number whatever.

[1] A logical system might be set up in which 0/0 would have this multiple-valued interpretation; such a system would apparently be about as convenient as the usual one; but it is not the usual one.

In what way, then, is the statement

$$\frac{2}{0} = \infty$$

to be interpreted? Strictly, in an arithmetical sense, it signifies merely that the operation $2/0$ is meaningless;—that the "result" of the operation is of one kind with the unicorn and the sea serpent. It is also used in a sense not exclusively arithmetical, generally with some sacrifice in clearness, to mean that "as x approaches zero, $2/x$ becomes infinite," or, in other words, "$2/x$ can be made as large as we please by taking x sufficiently small." It should be noted that even this interpretation implies no division by zero, but does imply division by many other numbers,—numbers near zero.

When we consider the fraction $(x^2 - 1)/(x - 1)$ for the value $x = 1$, we find that both numerator and denominator are entirely respectable numbers,—both are in fact zero; the fraction however is meaningless, since division by zero is a meaningless operation. When on the other hand we consider the fraction $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$, we find that even the numerator and denominator individually are meaningless.

But in both cases, when we speak of "evaluating the indeterminate form," we mean (or ought to mean) something which is unfortunately quite different from the literal implication of the words. We do not mean that we are to discover, by astute detective work, a "true value" which unkind fate is endeavoring to conceal from us. It is in no sense true that $(x^2 - 1)/(x - 1)$, which for $x = 1$ assumes the form $0/0$, really takes on for $x = 1$ the "true value" or "determinate value" 2. The value of $(x^2 - 1)/(x - 1)$ for $x = 1$ is not 2, does not exist, and cannot be brought into existence unless some other than the usual number system be used. What we can do, and what is often useful for us to do, is to find the limit approached by $(x^2 - 1)/(x - 1)$ as x approaches 1; this limit is indeed 2. If we write, as do some text-books:

"(1) $$\frac{x^2 - 1}{x - 1} = x + 1;$$

hence

(2) $$\frac{x^2 - 1}{x - 1} = 2 \qquad \text{when} \qquad x = 1,"$$

we are merely confusing the issue by a bald untruth. The following statement is true, and is the most that can be said without departing from the truth:

"(1) $$\frac{x^2 - 1}{x - 1} = x + 1 \qquad \text{when} \qquad x \neq 1;$$

hence

(2) $$\frac{x^2 - 1}{x - 1} \text{ approaches 2 as } x \text{ approaches 1."}$$

As indicated, (1) is true for $x \neq 1$. It is untrue for $x = 1$; the algebraic

operation by which it was obtained—formal division of the numerator by the denominator—cannot be performed when $x = 1$, although it can be performed for every other value of x.

Similarly, when we set out to "evaluate the indeterminate form $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ for $x = 1$," we must not think that we shall by sharpness of vision pierce an algebraic mist which hides from us the "true value" 1 of this expression for $x = 1$. We should mean that we are seeking the limit (if any exists) approached by $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ as x approaches 1; and we should mean nothing further than this.

For the value $x = 1$, $(x^2 - 1)/(x - 1)$ does assume the form $0/0$, and $(\tan \frac{1}{2}\pi x)/(\sec \frac{1}{2}\pi x)$ may be said to assume the form ∞/∞; but neither fraction has any *value* for the value of x in question, nor can we by any amount of ambiguous argument or inaccurate analogy force it to have such a value.

On The Geometrical Representation of Indeterminate Forms.

By R. E. Moritz, University of Washington.

Every teacher of the calculus will have witnessed the difficulty beginners have in properly interpreting the so-called indeterminate forms $0/0$, ∞/∞, $0\cdot\infty$, $\infty - \infty$, etc. This should cause no surprise; for the meaning of the symbols 0 and ∞, even when taken by themselves, is not too clear to the average student and the difficulty is, of course, accentuated when these symbols are combined as in the indeterminate forms.

It should be remembered in this connection that the indeterminate forms have not always been correctly interpreted even by mathematicians and that some of them remained subjects of controversy until comparatively recent times. The first complete evaluation of the form $0/0$ dates from John Bernoulli[1] (1704) and it was not until 1755 that Euler[2] gave the first published treatment of the forms ∞/∞, $0\cdot\infty$, and $\infty - \infty$. But the true meaning of 0^0 eluded even the great Euler, for he arrived at the conclusion[3] that 0^0 must always equal 1 by a process of reasoning which would show equally well that $0/0$ must always equal 1. Euler's erroneous conclusion was shared by many subsequent mathematicians for three quarters of a century. New proofs (!) of his conclusion appeared in creditable journals by eminent mathematicians as late as the middle third of the nineteenth century,[4] and this notwithstanding the great weight of the authority of Cauchy, who, in his *Cours d'Analyse,*[5] had given the interpretation of the exponential indeterminate forms which is universally accepted today.

In view of the fact that modern textbooks on the calculus generally avail themselves of geometrical illustrations to introduce new or difficult concepts, it

[1] *Opera*, vol. 1, pp. 401–405.

[2] *Opera Omnia*, series I, vol. 10, 1913, p. 576.

[3] Euler, *Opera Omnia*, series I, vol. 1, 1911, p. 65.

[4] Libri, G., Crelle's *Journal*, vol. 10, 1833, p. 303; Moebius, A. F., Crelle's *Journal*, vol. 12, 1834, p. 134.

[5] *Oeuvres*, 2d series, vol. 3, p. 70.